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The classical limit of quantum mechanics as a Lie algebra contraction

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Abstract. The classical limit of a quantum system with one degree of freedom is examined in terms of the contraction of the underlying non-compact kinematical algebra w_1 (the Weyl–Heisenberg algebra) to the three-dimensional Abelian algebra t_3. An appropriate definition of the contraction of a Lie algebra and of a sequence of its representations is given. For some quantum systems with simple explicitly integrable dynamics, it is shown how the classical Poisson bracket and classical trajectories are obtained in the limit. Each trajectory is associated with a one-dimensional representation of t_3 within a direct integral of such representations in a Hilbert space. Each of the corresponding generalized one-dimensional subspaces is stable under the action of the limiting dynamics, and a superselection rule arises naturally between any two such subspaces.

1. Introduction

The relationship between a quantum mechanical system and its classical counterpart, and the way in which the classical description can be obtained as a limiting form of the quantum mechanical one, have been topics of much discussion since the earliest days of quantum mechanics. Ehrenfest's work [1] is widely known, and most books on quantum mechanics include some discussion of this topic, often in terms of the WJKB approximation [2].

The present work is concerned with an approach to this question in Lie algebraic terms, with a central role being played by the contraction of representations of the relevant Lie algebras and their enveloping algebras [3]. It is well known that taking the classical limit, in the formal sense of Planck's constant $\hbar$ going to zero, can be viewed in terms of a contraction of the underlying kinematical algebra, the Weyl–Heisenberg algebra w_n for a system with n degrees of freedom, to an Abelian Lie algebra t_{2n+1} of the same dimension. However, there does not appear to be any specific discussion in the literature of the behaviour of the *representations* of w_n and t_{2n+1} involved in the taking of the classical limit, when viewed as a contraction. The aim of this paper is to present such a discussion, in the first instance for systems with a single degree of freedom, and a central role will be played in what follows by the contraction of a sequence of infinite-dimensional Hermitian matrix representations of w_1, to a direct integral of one-dimensional irreducible representations of t_3. It has been found

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necessary to employ a modified definition of the contraction of a Lie algebra and of its representations by the method of sequences of representations, those previously given [3, 4] being not quite suitable for this purpose.

An aspect of the classical limit that is not dealt with very directly in other approaches, is the way in which the Hilbert space structure of the quantum formalism is connected to the phase space structure of the classical formalism, with the principal objects of interest being linear operators on Hilbert spaces in the former, and functions on phase spaces in the latter. In the present approach, the apparatus of Lie algebra generators (operators), acting on a Hilbert space, inevitably survives in the contraction limit. However, this apparatus, considered in the Heisenberg picture of quantum mechanics, can be related in a very direct way to the usual description of a classical dynamical system in terms of canonical position and momentum variables q, p, trajectories in phase space, Poisson brackets, etc, mainly because the quantum mechanical operators q and p commute in the contraction limit and are diagonal on the one-dimensional irreducible representations of t_3, and also because a superselection rule arises between any two of the one-dimensional limiting representations of t_3 and, at least for some simple model systems considered, this superselection rule is stable under the dynamics of the system. The Hilbert space becomes in effect a union of one-dimensional subspaces, corresponding to the points in phase space [5].

Although only quantum mechanical systems with a single degree of freedom are considered in this paper, there is a natural and straightforward generalization of the ideas involved to systems with n degrees of freedom. On the other hand, it has not yet been possible to push those ideas through to completion for any non-integrable system, and that should certainly be the subject of future investigation. Such systems have been the focus of much attention in recent discussions of the classical limit [6] In what follows, recovery of classical trajectories is demonstrated for linear systems, and also for an explicitly integrable nonlinear system which has been discussed in the literature [7–9] as one exemplifying some differences between quantum mechanics and classical mechanics.

2. Contraction of Lie algebras and their representations

There have been several definitions in the literature [3, 4] of contractions of Lie algebras and Lie groups, and of their representations, but none is quite suitable for the purpose here. Accordingly, a slightly modified definition is first presented of the contraction of a Lie algebra, and of the contraction limit of a sequence of its representations.

Definition 2.1. Contraction of a Lie algebra
Let $\mathcal{G}$ be a real Lie algebra with generators X_i, $i = 1, \ldots, N$, satisfying the bracket relations $[X_i, X_j] = \sum_{k=1}^{N} c_{ij}^k X_k$, where the c_{ij}^k are structure constants of the algebra. Consider new generators of the form $X_i^\epsilon = \epsilon^{\alpha_i} X_i$, $\epsilon > 0$, $\alpha_i \geqslant 0$, $(i = 1, \ldots, N)$, the α_i being not all zero, and such that for all $i, j, k = 1, \ldots, N$, $\alpha_i + \alpha_j - \alpha_k \geqslant 0$ whenever $c_{ij}^k \neq 0$. These X_i^ϵ satisfy the relations

$$[X_i^\epsilon, X_j^\epsilon] = \sum_{k=1}^{N} \epsilon^{\alpha_i + \alpha_j - \alpha_k} c_{ij}^k X_k \tag{1}$$

The Lie algebra $\mathcal{G}^*$ that results from a contraction of $\mathcal{G}$ with given α_i has generators Y_i, with

$$[Y_i, Y_j] = \sum_{k=1}^{N} C_{ij}^k Y_k \qquad C_{ij}^k = \lim_{\epsilon \to 0} \epsilon^{\alpha_i + \alpha_j - \alpha_k} c_{ij}^k \tag{2}$$

Remarks

(1) The ϵ appearing in this definition will be called the contraction parameter and the α_i will be called the contraction indices. It is not necessary for the contraction indices to be non-negative integers.

(2) There is always at least one set of contraction indices α_i satisfying the condition $\alpha_i + \alpha_j - \alpha_k \geqslant 0$ for an arbitrary Lie algebra $\mathcal{G}$. For example, by setting $\alpha_i \equiv \alpha > 0$ the contraction to the Abelian Lie algebra t_N of dimension N is obtained.

(3) The Lie algebra $\mathcal{G}^*$ obtained by a contraction is always non-compact [4] even though $\mathcal{G}$ may be either compact or non-compact.

(4) Different choices of the α_i can lead to the same contracted algebra.

Definition 2.2. Contraction limit of a sequence of representations of a Lie algebra
Consider the contraction of one Lie algebra to another ($\mathcal{G} \to \mathcal{G}^*$) as defined above.

(1) For $m = 1, 2, ...,$

(a) Let $\mathcal{H}_m$ denote a Hilbert space with inner product $(\cdot,\cdot)_m$, and let $\{\phi_{(m)r}\}_{r=0}^{\infty}$ denote a basis in $\mathcal{H}_m$ orthonormal with respect to that inner product. Let $\mathcal{S}_m$ denote the linear span of this set of vectors (that is, $\mathcal{S}_m$ is the dense subspace of $\mathcal{H}_m$ consisting of all finite linear combinations of the vectors $\phi_{(m)r}$, $r = 0, 1, \ldots$).

(b) Let π_m denote a representation of $\mathcal{G}$ on $\mathcal{S}_m$ (so that $\mathcal{S}_m$ is a common invariant dense domain for the operators $\pi_m(X_i)$, $i = 1, \ldots, N$).

(c) Set $[\pi_m(X_i)]_{rs} = (\phi_{(m)r}, \pi_m(X_i)\,\phi_{(m)s})_m$ for $i = 1, \ldots, N$ and all non-negative integers r, s.

(d) Suppose that $\pi_m(X_i)$ is column-finite. More precisely, suppose that, for each non-negative integer s, $[\pi_m(X_i)]_{rs} = 0$, for $r > K_s$, where K_s may depend on s but must be independent of m.

(2) (a) Let $\mathcal{H}_\infty$ denote a Hilbert space with inner product $(\cdot,\cdot)_\infty$. Let $\{\phi_{(\infty)r}\}_{r=0}^{\infty}$ denote a basis in $\mathcal{H}_\infty$, orthonormal with respect to $(\cdot,\cdot)_\infty$ and let $\mathcal{S}_\infty$ denote the linear span of these vectors.

(b) Let π_∞ denote a representation of the Lie algebra $\mathcal{G}^*$ on $\mathcal{S}_\infty$ where $\mathcal{G}^*$ is the contraction of the Lie algebra $\mathcal{G}$ with contraction indices α_i defined as above.

(c) Set $[\pi_\infty(Y_i)]_{rs} = (\phi_{(\infty)r}, \pi_\infty(Y_i)\phi_{(\infty)s})_\infty$ for $i = 1, \ldots, N$; and all non-negative integers r, s, where the Y_i are generators of $\mathcal{G}^*$.

(3) Let $\{\epsilon_m\}_{m=1}^{\infty}$ denote a sequence of real positive numbers such that $\lim_{m \to \infty} \epsilon_m = 0$.
The representation π_∞ is called a contraction limit of the sequence of representations $\{\pi_m\}_{m=1}^{\infty}$ if

$$\lim_{m \to \infty} (\epsilon_m)^{\alpha_i} [\pi_m(X_i)]_{rs} = [\pi_\infty(Y_i)]_{rs} \tag{3}$$

for $i = 1, \ldots, N$ and all non-negative integers r, s.

Remarks (1) The contraction limit can depend on the choice of bases in S_m, $m = 1,2,\ldots$. This may appear to be a deficiency of the definition, but as will be seen below, the consequent flexibility is important

(2) The condition 1(d) is included to guarantee that

$$\lim_{m\to\infty} \sum_{q,r,\ldots,s=0}^{\infty} (\epsilon_m)^{\alpha_i+\alpha_j+\cdots+\alpha_k}[\pi_m(X_i)]_{pq}[\pi_m(X_j)]_{qr}\cdots[\pi_m(X_k)]_{st}$$

$$= \sum_{q,r,\ldots,s=0}^{\infty} [\pi_\infty(Y_i)]_{pq}[\pi_\infty(Y_j)]_{qr}\cdots[\pi_\infty(Y_k)]_{st} \tag{4}$$

for arbitrarily large but finite products of the generators. In effect, this condition ensures that the sequence of representations of the enveloping algebra of $\mathcal{G}$ (which is the algebra spanned by linear combinations of arbitrary finite products of the generators) is contracted to a representation of the enveloping algebra of $\mathcal{G}^*$.

(3) In a particular case, $[\pi_m(X_i)]_{rs}$ may be non-zero only for $r,s \leqslant N_m < \infty$. This definition can therefore accommodate the case of a sequence of finite-dimensional representations of increasing dimension being contracted to an infinite-dimension representation. It is not necessary that the representations π_m or the representation π_∞ be irreducible.

(4) This definition does not preclude the possibility that the representation of $\mathcal{G}^*$ obtained in the contraction limit may be unfaithful; in particular, that $[\pi_\infty(Y_i)]_{rs}$ may vanish for all r,s, for some i

3. Contraction of representations of w_1 to representations of t_3

Let $\bar{a}, a$ be a pair of boson creation and annihilation operators. By that is meant in particular that

(1) there is a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ invariant under the action of the boson operators, on which a and $\bar{a}$ are Hermitian conjugate to each other, and on which $[a,\bar{a}] = \mathbb{I}$ where $\mathbb{I}$ is the identity operator on $\mathcal{H}$; and

(2) $\mathcal{D}$ contains a 'vacuum vector' ξ_0, normalized with respect to the inner product $(\cdot,\cdot)$ on $\mathcal{H}$ and such that $a\xi_0 = 0$. Consequently, it also contains the orthonormal vectors $\xi_r = (1/\sqrt{r!})\bar{a}^r\xi_0$ for $r = 0,1,\ldots$, which are assumed to form a basis in $\mathcal{H}$.

The action of the boson operators on these vectors is given by the equations

$$a\xi_r = \sqrt{r}\,\xi_{r-1} \qquad \bar{a}\xi_r = \sqrt{r+1}\,\xi_{r+1} \tag{5}$$

and so ξ_r is an eigenvector of the number operator $N = \bar{a}a$ with eigenvalue r.

The Lie algebra w_1 has three generators X_1, X_2, X_3, which satisfy the bracket relations

$$[X_1,X_2] = \mathrm{i}\hbar X_3 \qquad [X_1,X_3] = [X_2,X_3] = 0. \tag{6}$$

In quantum mechanics, the generators X_1, X_2 are associated with position and momentum variables, respectively, and $\hbar$ is the modified Planck constant. More precisely, quantum mechanics utilizes the boson representation of w_1 (or one equivalent to it) which can be defined on $\mathcal{D}$ by setting

$$\begin{aligned} Q &= \pi(X_1) = \lambda(a+\bar{a})/\sqrt{2} \\ P &= \pi(X_2) = -\mathrm{i}\kappa(a-\bar{a})/\sqrt{2} \\ I &= \pi(X_3) = \mathbb{I} \end{aligned} \tag{7}$$

where λ and κ are fixed length and momentum scales respectively satisfying $\lambda\kappa = \hbar$. These representatives are Hermitian on $\mathcal{D}$ with respect to $(\ ,\cdot)$, and their matrix elements are determined from (5) by the equations

$$\begin{aligned} Q\xi_r &= \frac{\lambda}{\sqrt{2}}(\sqrt{r}\,\xi_{r-1} + \sqrt{r+1}\,\xi_{r+1}) \\ P\xi_r &= \frac{-\mathrm{i}\kappa}{\sqrt{2}}(\sqrt{r}\,\xi_{r-1} - \sqrt{r+1}\,\xi_{r+1}) \\ I\xi_r &= \xi_r \end{aligned} \tag{8}$$

The Lie algebra w_1 is contracted to t_3 by setting $X_1^\epsilon = \epsilon X_1$, $X_2^\epsilon = \epsilon X_2$, $X_3^\epsilon = X_3$, where ϵ is the contraction parameter The bracket relations satisfied by the contracting generators are then

$$[X_1^\epsilon, X_2^\epsilon] = \mathrm{i}\epsilon^2\hbar X_3^\epsilon \qquad [X_1^\epsilon, X_3^\epsilon] = [X_2^\epsilon, X_3^\epsilon] = 0 \tag{9}$$

and the first of these formally vanishes in the contraction limit, so that the bracket relations reduce to those defining t_3, that is, $[Y_i, Y_j] = 0$ for $i, j = 1, 2, 3$.

The following result is the key to the subsequent discussion.

Theorem. Choose a dimensionless constant $\zeta \geqslant 0$ Then there is a sequence of matrix representations $\{\pi_m\}_{m=1}^{\infty}$ of w_1, each equivalent to the boson representation π of (7) and acting on a subspace $\mathcal{S}_m$ of $\mathcal{H}_m$, whose contraction limit is a representation π_∞ of t_3 acting on a subspace $\mathcal{S}_\infty$ of $\mathcal{H}_\infty$. This representation is a direct integral of irreducible one-dimensional Hermitian representations with

$$\begin{aligned} Q_\infty &= \lambda \int_0^{2\pi} \oplus[\sqrt{2\zeta}\,\cos\theta]\,\mathrm{d}\theta \\ P_\infty &= \kappa \int_0^{2\pi} \oplus[\sqrt{2\zeta}\,\sin\theta]\,\mathrm{d}\theta \\ I_\infty &= \int_0^{2\pi} \oplus[1]\,\mathrm{d}\theta \end{aligned} \tag{10}$$

and

$$Q_\infty^2/\lambda^2 + P_\infty^2/\kappa^2 = 2\zeta I_\infty \tag{11}$$

where

$$Q_\infty = \pi_\infty(Y_1) \qquad P_\infty = \pi_\infty(Y_2) \qquad I_\infty = \pi_\infty(Y_3) \tag{12}$$

Proof Let $[\alpha]$ denote the integer part of the real number α Consider a sequence of equivalent representations π_m of w_1, labelled by $m = 1, 2, \ldots$, where for each m

$$\pi_m(X_1) = Q \qquad \pi_m(X_2) = P \qquad \pi_m(X_3) = I \tag{13}$$

as in (8), with the basis $\{\phi_{(m)r}\}_{r=0}^{\infty}$ of $\mathcal{H}_m$ described by

$$\begin{aligned} &\phi_{(m)2r} = \xi_{M_m - r} && r = 0, 1, 2, \ldots, M_m \\ &\phi_{(m)2r-1} = \xi_{M_m + r} && r = 1, 2, \ldots, M_m \\ &\phi_{(m)r} = \xi_r && r > 2M_m \end{aligned} \tag{14}$$

where $M_m = [\![\zeta m]\!]$ For each m, the value of ϵ_m is chosen to be $1/\sqrt{m}$ It is convenient to set $Q_m = \epsilon_m Q$, $P_m = \epsilon_m P$, $I_m = I$. Matrix elements of Q_m, P_m, I_m follow from (8), (13) and (14), which imply that

$$I_m \phi_{(m)r} = \phi_{(m)r} \qquad r = 0, 1, 2, \ldots$$

$$Q_m \phi_{(m)0} = \frac{\lambda}{\sqrt{2m}} (\sqrt{M_m}\, \phi_{(m)2} + \sqrt{M_m + 1}\, \phi_{(m)1})$$

$$P_m \phi_{(m)0} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}} (\sqrt{M_m}\, \phi_{(m)2} - \sqrt{M_m + 1}\, \phi_{(m)1})$$

$$Q_m \phi_{(m)1} = \frac{\lambda}{\sqrt{2m}} (\sqrt{M_m + 1}\, \phi_{(m)0} + \sqrt{M_m + 2}\, \phi_{(m)3})$$

$$P_m \phi_{(m)1} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}} (\sqrt{M_m + 1}\, \phi_{(m)0} - \sqrt{M_m + 2}\, \phi_{(m)3})$$

$$Q_m \phi_{(m)2r} = \frac{\lambda}{\sqrt{2m}} (\sqrt{M_m - r}\, \phi_{(m)2r+2} + \sqrt{M_m - r + 1}\, \phi_{(m)2r-2}) \qquad r = 1, 2, \ldots, M_m - 1$$

$$P_m \phi_{(m)2r} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}} (\sqrt{M_m - r}\, \phi_{(m)2r+2} - \sqrt{M_m - r + 1}\, \phi_{(m)2r-2}) \qquad r = 1, 2, \ldots, M_m - 1$$

$$Q_m \phi_{(m)2r-1} = \frac{\lambda}{\sqrt{2m}} (\sqrt{M_m + r}\, \phi_{(m)2r-3} + \sqrt{M_m + r + 1}\, \phi_{(m)2r+1}) \qquad r = 2, 3, \ldots, M_m$$

$$P_m \phi_{(m)2r-1} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}} (\sqrt{M_m + r}\, \phi_{(m)2r-3} - \sqrt{M_m + r + 1}\, \phi_{(m)2r+1}) \qquad r = 2, 3, \ldots, M_m$$

$$Q_m \phi_{(m)2M_m} = \frac{\lambda}{\sqrt{2m}} \phi_{(m)2M_m - 2}$$

$$P_m \phi_{(m)2M_m} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}} \phi_{(m)2M_m - 2}$$

$$Q_m \phi_{(m)2M_m+1} = \frac{\lambda}{\sqrt{2m}} (\sqrt{2M_m + 1}\, \phi_{(m)2M_m-1} + \sqrt{2M_m + 2}\, \phi_{(m)2M_m+2})$$

$$P_m\phi_{(m)2M_m+1} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}}(\sqrt{2M_m+1}\,\phi_{(m)2M_m-1} - \sqrt{2M_m+2}\,\phi_{(m)2M_m+2})$$

$$Q_m\phi_{(m)r} = \frac{\lambda}{\sqrt{2m}}(\sqrt{r}\,\phi_{(m)r-1} + \sqrt{r+1}\,\phi_{(m)r+1}) \qquad r > 2M_m + 1$$

$$P_m\phi_{(m)r} = \frac{-\mathrm{i}\kappa}{\sqrt{2m}}(\sqrt{r}\,\phi_{(m)r-1} - \sqrt{r+1}\,\phi_{(m)r+1}) \qquad r > 2M_m + 1. \tag{15}$$

For example, the matrix of Q_m is given in table 1.

The matrix elements of the limiting operators Q_∞, P_∞, I_∞ of (12), expressed on the Hilbert space $\mathcal{H}_\infty$ with basis vectors $\{\phi_{(\infty)r}\}_{r=0}^{\infty}$ are found from (15) to be

$$\begin{aligned}
&I_\infty\phi_{(\infty)r} = \phi_{(\infty)r} \qquad r = 0,1,2,\ldots\\
&Q_\infty\phi_{(\infty)0} = \lambda\sqrt{\zeta/2}\,(\phi_{(\infty)2} + \phi_{(\infty)1})\\
&P_\infty\phi_{(\infty)0} = -\mathrm{i}\kappa\sqrt{\zeta/2}\,(\phi_{(\infty)2} - \phi_{(\infty)1})\\
&Q_\infty\phi_{(\infty)1} = \lambda\sqrt{\zeta/2}\,(\phi_{(\infty)0} + \phi_{(\infty)3})\\
&P_\infty\phi_{(\infty)1} = -\mathrm{i}\kappa\sqrt{\zeta/2}\,(\phi_{(\infty)0} - \phi_{(\infty)3})\\
&Q_\infty\phi_{(\infty)2r} = \lambda\sqrt{\zeta/2}\,(\phi_{(\infty)2r+2} + \phi_{(\infty)2r-2}) \qquad r = 1,2,\ldots\\
&P_\infty\phi_{(\infty)2r} = -\mathrm{i}\kappa\sqrt{\zeta/2}\,(\phi_{(\infty)2r+2} - \phi_{(\infty)2r-2}) \qquad r = 1,2,\ldots\\
&Q_\infty\phi_{(\infty)2r-1} = \lambda\sqrt{\zeta/2}\,(\phi_{(\infty)2r-3} + \phi_{(\infty)2r+1}) \qquad r = 2,3,\ldots\\
&P_\infty\phi_{(\infty)2r-1} = -\mathrm{i}\kappa\sqrt{\zeta/2}\,(\phi_{(\infty)2r-3} - \phi_{(\infty)2r+1}) \qquad r = 2,3,\ldots
\end{aligned} \tag{16}$$

For example, the matrix of Q_∞ is given in table 2.

It can then be checked that on the linear span $\mathcal{S}_\infty$ of these basis vectors

$$[Q_\infty, P_\infty] = [Q_\infty, I_\infty] = [P_\infty, I_\infty] = 0 \tag{17}$$

defining a representation π_∞ of t_3. There is an equivalent and more compact way of describing the representations π_m and π_∞. This is done by rewriting the basis of $\mathcal{H}_m$ in the form

$$\{\psi_{(m)r} | \psi_{(m)r} = \xi_{r+M_m}, r \geqslant -M_m\} \tag{18}$$

and the basis of $\mathcal{H}_\infty$ in the form

$$\{\psi_{(\infty)r} | \psi_{(\infty)-r} = \phi_{(\infty)2r}, r = 0,1,2,\ldots; \psi_{(\infty)r} = \phi_{(\infty)2r-1}, r = 1,2,\ldots\}. \tag{19}$$

Then

$$\begin{aligned}
&Q_m\psi_{(m)r} = \frac{\lambda}{\sqrt{2m}}(\sqrt{M_m+r}\,\psi_{(m)r-1} + \sqrt{M_m+r+1}\,\psi_{(m)r+1})\\
&P_m\psi_{(m)r} = -\frac{\mathrm{i}\kappa}{\sqrt{2m}}(\sqrt{M_m+r}\,\psi_{(m)r-1} - \sqrt{M_m+r+1}\,\psi_{(m)r+1})\\
&I_m\psi_{(m)r} = \psi_{(m)r}
\end{aligned} \tag{20}$$

Table 1. Matrix of Q_m in the chosen basis. As $m \to \infty$, $M_m/m \to \zeta$

$$Q_m = \frac{\lambda}{\sqrt{2m}} \left(\begin{array}{cccccccccccccc}
0 & \sqrt{M_m+1} & \sqrt{M_m} & 0 & 0 & 0 & & & & & & & & \\
\sqrt{M_m+1} & 0 & 0 & \sqrt{M_m+2} & 0 & 0 & & & & & & & & \\
\sqrt{M_m} & 0 & 0 & 0 & \sqrt{M_m-1} & 0 & & & & & & & & \\
0 & \sqrt{M_m+2} & 0 & 0 & 0 & \sqrt{M_m+3} & & & & & & & & \\
0 & 0 & \sqrt{M_m-1} & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & \sqrt{M_m+3} & 0 & 0 & \ddots & & & & & & & \\
 & & & & & & & \ddots & & & & & & \\
 & & & & & & & & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & \sqrt{2M_m+1} & 0 & 0 \\
 & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & \sqrt{2M_m+1} & 0 & 0 & \sqrt{2M_m+2} & 0 \\
 & & & & & & & & 0 & 0 & 0 & \sqrt{2M_m+2} & 0 & \sqrt{2M_m+3} \\
 & & & & & & & & 0 & 0 & 0 & 0 & \sqrt{2M_m+3} & 0
\end{array} \right)$$

for all integers $r \geqslant -M_m$, and in the contraction limit

$$
\begin{aligned}
Q_\infty \psi_{(\infty)r} &= \lambda\sqrt{\zeta/2}\,(\psi_{(\infty)r-1} + \psi_{(\infty)r+1}) \\
P_\infty \psi_{(\infty)r} &= -\mathrm{i}\kappa\sqrt{\zeta/2}\,(\psi_{(\infty)r-1} - \psi_{(\infty)r+1}) \\
I_\infty \psi_{(\infty)r} &= \psi_{(\infty)r}
\end{aligned}
\tag{21}
$$

for all integers r. It is more easily checked that (17) hold on this set of basis vectors. It is also noted from (21) that

$$
(Q_\infty^2/\lambda^2 + P_\infty^2/\kappa^2)\psi_{(\infty)r} = 2\zeta\psi_{(\infty)r}
\tag{22}
$$

and thus (11) holds for the representation of t_3 obtained here. Appropriate minor adjustments have to be made, here and below, to accommodate the special case $\zeta = 0$.

It can be seen as follows that the representation of t_3 obtained here is a direct integral of one-dimensional Hermitian representations.

Let $(\ ,\cdot)_\infty$ be the inner product of the Hilbert space $\mathcal{H}_\infty$ with respect to which the $\psi_{(\infty)r}$ are orthonormal for all integers r, and let $\mathbb{I}_\infty$ be the identity operator on $\mathcal{H}_\infty$. Also introduce the dual space $\mathcal{T}_\infty$ of $\mathcal{S}_\infty$, whose elements (linear functionals on $\mathcal{S}_\infty$) are associated with all formal vectors of the form $\sum_{r=-\infty}^{\infty} c_r \psi_{(\infty)r}$, where the c_r are arbitrary complex numbers. The commuting operators Q_∞, P_∞, I_∞ have common generalized eigenvectors in $\mathcal{T}_\infty$. They have the form

$$
\Phi_\infty(\zeta,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{r=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}r\theta}\psi_{(\infty)r} \qquad 0 \leqslant \theta < 2\pi
\tag{23}
$$

and they can be seen from (21) to satisfy

$$
\begin{aligned}
Q_\infty \Phi_\infty(\zeta,\theta) &= \lambda\sqrt{2\zeta}\cos\theta\,\Phi_\infty(\zeta,\theta) \\
P_\infty \Phi_\infty(\zeta,\theta) &= \kappa\sqrt{2\zeta}\sin\theta\,\Phi_\infty(\zeta,\theta) \\
I_\infty \Phi_\infty(\zeta,\theta) &= \Phi_\infty(\zeta,\theta)
\end{aligned}
\tag{24}
$$

in the sense that

$$
\begin{aligned}
(\Phi_\infty(\zeta,\theta), Q_\infty\psi_{(\infty)r})_\infty &= \lambda\sqrt{2\zeta}\cos\theta(\Phi_\infty(\zeta,\theta),\psi_{(\infty)r})_\infty \\
(\Phi_\infty(\zeta,\theta), P_\infty\psi_{(\infty)r})_\infty &= \kappa\sqrt{2\zeta}\sin\theta(\Phi_\infty(\zeta,\theta),\psi_{(\infty)r})_\infty \\
(\Phi_\infty(\zeta,\theta), I_\infty\psi_{(\infty)r})_\infty &= (\Phi_\infty(\zeta,\theta),\psi_{(\infty)r})_\infty
\end{aligned}
\tag{25}
$$

for all r. Note the consistency of these equations with (22). These generalized eigenvectors have a 'delta function normalization'

$$
(\Phi_\infty(\zeta,\theta), \Phi_\infty(\zeta,\theta'))_\infty = \delta(\theta-\theta')
\tag{26}
$$

for $0 \leqslant \theta,\theta' < 2\pi$. It can be shown that the vectors $\Phi_\infty(\zeta,\theta)$ form a (continuous) basis for the Hilbert space $\mathcal{H}_\infty$ spanned by the vectors $\psi_{(\infty)r}$; in particular

$$
\psi_{(\infty)r} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}r\theta}\Phi_\infty(\zeta,\theta)\,\mathrm{d}\theta.
\tag{27}
$$

This completes the proof of the theorem. □

The reducible (direct integral) representation of t_3 obtained here in the contraction limit is effectively parameterized by the real number ζ. The possibility of obtaining direct sum combinations of representations of this type, corresponding to a range of ζ values, is discussed in the appendix where a method of construction of such representations is outlined.

Table 2. Matrix of Q_∞.

$$Q_\infty = \lambda\sqrt{\tfrac{\zeta}{2}} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\ & & & & & & & \ddots \end{pmatrix}.$$

4. The classical limit of a quantum system with a single degree of freedom

Consider now a quantum system with a single degree of freedom, kinematical algebra w_1, *and a time-independent Hamiltonian operator* H. *The representation of* w_1 *is* spanned by operators Q, P, I as in (7), acting on $\mathcal{H}$

Suppose H can be written as a polynomial in Q and P. More precisely, suppose it can be written in the form

$$H = \sum_{\substack{0\leqslant m_1, m_2 \\ m_1+m_2\leqslant N}} a_{m_1 m_2} \tfrac{1}{2}(Q^{m_1}P^{m_2} + P^{m_2}Q^{m_1}) \tag{28}$$

where the $a_{m_1 m_2}$ are real constants of the appropriate dimensions and N is some positive integer. In this form, the Hamiltonian operator is Hermitian with respect to $(\cdot,\cdot)$, it will be supposed that H is also self-adjoint with respect to this inner product. More general terms in H of the form $Q^{m_1}P^{n_1}Q^{m_2}P^{n_2}\ldots$ could be considered, but this would not substantially alter the discussion

The time dependence of the quantum operators in the Heisenberg picture is given by the operator differential equations

$$\begin{aligned} \dot{Q} &= \frac{1}{i\hbar}[Q,H] = \sum_{\substack{0\leqslant m_1, m_2 \\ m_1+m_2\leqslant N}} \frac{m_2}{2} a_{m_1 m_2}(Q^{m_1}P^{m_2-1} + P^{m_2-1}Q^{m_1}) \\ P &= \frac{1}{i\hbar}[P,H] = -\sum_{\substack{0\leqslant m_1, m_2 \\ m_1+m_2\leqslant N}} \frac{m_1}{2} a_{m_1 m_2}(Q^{m_1-1}P^{m_2} + P^{m_2}Q^{m_1-1}). \end{aligned} \tag{29}$$

Formal solutions to these equations have the form

$$\begin{aligned} Q(t) &= U(t)^\dagger Q(0) U(t) \qquad & P(t) &= U(t)^\dagger P(0) U(t) \\ I(t) &\equiv I(0) & U(t) &= \mathrm{e}^{H(Q(0),P(0))t/i\hbar} \end{aligned} \tag{30}$$

where $U(t)$ is the unitary evolution operator.

The corresponding classical Hamiltonian system has a Hamiltonian function

$$H = \sum_{\substack{0\leqslant m_1, m_2 \\ m_1+m_2\leqslant N}} a_{m_1 m_2} q^{m_1} p^{m_2} \tag{31}$$

and dynamical equations

$$\dot{q} = \frac{\partial H}{\partial p} = \sum_{\substack{0 \leqslant m_1, m_2 \\ m_1 + m_2 \leqslant N}} m_2 a_{m_1 m_2} q^{m_1} p^{m_2 - 1}$$
$$\dot{p} = -\frac{\partial H}{\partial q} = -\sum_{\substack{0 \leqslant m_1, m_2 \\ m_1 + m_2 \leqslant N}} m_1 a_{m_1 m_2} q^{m_1 - 1} p^{m_2}. \tag{32}$$

Let (q_0, p_0) be an initial condition for a classical trajectory $(q(t), p(t))$ of (32) and set $q_0 = \lambda\sqrt{2\zeta}\cos\theta$, $p_0 = \kappa\sqrt{2\zeta}\sin\theta$ for $\zeta = [(q_0/\lambda)^2 + (p_0/\kappa)^2]/2 \geqslant 0$ and $0 \leqslant \theta < 2\pi$. Here, λ, κ are the scales introduced in the previous section Note that the scale of $|q_0 p_0|$ can be made macroscopic, or large compared with $\hbar$, by choosing ζ sufficiently large.

Consider now the contraction of w_1 to t_3 given in the previous section, where the number states ξ_n are eigenvectors of $\bar{a}(0)a(0)$, and $a(0)$ and $\bar{a}(0)$ are related to $Q(0)$ and $P(0)$ as in (7). Again set $Q_m = \epsilon_m Q$, $P_m = \epsilon_m P$, $I_m = I$, and let Q_m, P_m, I_m have the representation (20), at time $t = 0$. Then from (20)

$$Q_m(0)\psi_{(m)r} = \frac{\lambda}{\sqrt{2m}}\left(\sqrt{M_m + r}\,\psi_{(m)r-1} + \sqrt{M_m + r + 1}\,\psi_{(m)r+1}\right)$$
$$P_m(0)\psi_{(m)r} = \frac{-i\kappa}{\sqrt{2m}}\left(\sqrt{M_m + r}\,\psi_{(m)r-1} - \sqrt{M_m + r + 1}\,\psi_{(m)r+1}\right)$$
$$I_m(0)\psi_{(m)r} = \psi_{(m)r} \tag{33}$$

and

$$[(Q_m(0)/\lambda)^2 + (P_m(0)/\kappa)^2]\psi_{(m)r} = [(2M_m + 2r + 1)/m]\psi_{(m)r} \tag{34}$$

for $r \geqslant -M_m$. The action of $Q_\infty(0)$, $P_\infty(0)$ and $I_\infty(0)$ on the vectors $\psi_{(\infty)r}$ for all integers r is given by (21), where $Q_\infty(0)$, $P_\infty(0)$ and $I_\infty(0)$ are the operators obtained in the contraction limit. On the generalized eigenvectors $\Phi_\infty(\zeta, \theta)$,

$$Q_\infty(0)\Phi_\infty(\zeta,\theta) = q_0\Phi_\infty(\zeta,\theta)$$
$$P_\infty(0)\Phi_\infty(\zeta,\theta) = p_0\Phi_\infty(\zeta,\theta) \tag{35}$$
$$I_\infty(0)\Phi_\infty(\zeta,\theta) = \Phi_\infty(\zeta,\theta)$$

in the sense described in (25).

The time evolution of the contracting operators $Q_m(t)$, $P_m(t)$, $I_m(t)$ is taken to be determined by the operator differential equations

$$\dot{Q}_m(t) = \frac{1}{i\epsilon_m^2\hbar}[Q_m(t), H_m]$$
$$= \sum_{\substack{0 \leqslant m_1, m_2 \\ m_1 + m_2 \leqslant N}} \frac{m_2}{2} a_{m_1 m_2}[Q_m(t)^{m_1} P_m(t)^{m_2 - 1} + P_m(t)^{m_2 - 1} Q_m(t)^{m_1}]$$
$$\dot{P}_m(t) = \frac{1}{i\epsilon_m^2\hbar}[P_m(t), H_m] \tag{36}$$
$$= -\sum_{\substack{0 \leqslant m_1, m_2 \\ m_1 + m_2 \leqslant N}} \frac{m_1}{2} a_{m_1 m_2}[Q_m(t)^{m_1 - 1} P_m(t)^{m_2} + P_m(t)^{m_2} Q_m(t)^{m_1 - 1}]$$
$$\dot{I}_m(t) = \frac{1}{i\epsilon_m^2\hbar}[I_m(t), H_m] \equiv 0$$

where

$$
\begin{aligned}
H_m &\equiv H(Q_m(0), P_m(0)) \\
&= \sum_{\substack{0 \leqslant m_1, m_2 \\ m_1 + m_2 \leqslant N}} \tfrac{1}{2} a_{m_1 m_2} (Q_m(0)^{m_1} P_m(0)^{m_2} + P_m(0)^{m_2} Q_m(0)^{m_1})
\end{aligned} \tag{37}
$$

Formal solutions in terms of a unitary evolution operator $U_m(t)$ can be given in the form

$$
\begin{aligned}
&Q_m(t) = U_m(t)^\dagger Q_m(0) U_m(t) \qquad && P_m(t) = U_m(t)^\dagger P_m(0) U_m(t) \\
&I_m(t) \equiv I_m(0) && U_m(t) = \mathrm{e}^{H_m t / \mathrm{i}\epsilon_m^2 \hbar}.
\end{aligned} \tag{38}
$$

The power of ϵ_m appearing in (36) and (38) is determined by the consideration that (36) should in general not diverge or vanish as $\epsilon_m \to 0$.

Let $(\cdot, \cdot)_m$ be the inner product of the Hilbert space $\mathcal{H}_m$ with respect to which the $\psi_{(m)r}$ are orthonormal for integers $r \geqslant -M_m$. Let $\mathcal{S}_m$ denote the linear span of these vectors and let $\mathcal{T}_m$ denote its dual space which is associated with all formal vectors of the form $\sum_{r=-M_m}^{\infty} c_r \psi_{(m)r}$ where the c_r are complex numbers. Let $\mathbb{I}_m$ denote the identity operator on $\mathcal{H}_m$ and set

$$
\Phi_m(\zeta, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{r=-M_m}^{\infty} \mathrm{e}^{\mathrm{i}r\theta} \psi_{(m)r}. \tag{39}
$$

From the definitions (39) and (23) of $\Phi_m(\zeta, \theta)$ and $\Phi_\infty(\zeta, \theta)$ respectively, and the definition of the contraction of a sequence of representations, it follows that

$$
\begin{aligned}
&\lim_{m\to\infty} Q_m(0) \Phi_m(\zeta, \theta) = Q_\infty(0) \Phi_\infty(\zeta, \theta) = q_0 \Phi_\infty(\zeta, \theta) \\
&\lim_{m\to\infty} P_m(0) \Phi_m(\zeta, \theta) = P_\infty(0) \Phi_\infty(\zeta, \theta) = p_0 \Phi_\infty(\zeta, \theta) \\
&\lim_{m\to\infty} I_m(0) \Phi_m(\zeta, \theta) = I_\infty(0) \Phi_\infty(\zeta, \theta) = \Phi_\infty(\zeta, \theta)
\end{aligned} \tag{40}
$$

in the sense that

$$
\begin{aligned}
&\lim_{m\to\infty} (\Phi_m(\zeta, \theta), Q_m(0)\psi_{(m)r})_m = (\Phi_\infty(\zeta, \theta), Q_\infty(0)\psi_{(\infty)r})_\infty = q_0 (\Phi_\infty(\zeta, \theta), \psi_{(\infty)r})_\infty \\
&\lim_{m\to\infty} (\Phi_m(\zeta, \theta), P_m(0)\psi_{(m)r})_m = (\Phi_\infty(\zeta, \theta), P_\infty(0)\psi_{(\infty)r})_\infty = p_0 (\Phi_\infty(\zeta, \theta), \psi_{(\infty)r})_\infty \\
&\lim_{m\to\infty} (\Phi_m(\zeta, \theta), I_m(0)\psi_{(m)r})_m = (\Phi_\infty(\zeta, \theta), I_\infty(0)\psi_{(\infty)r})_\infty = (\Phi_\infty(\zeta, \theta), \psi_{(\infty)r})_\infty
\end{aligned} \tag{41}
$$

for all r. This result extends to any finite polynomial $A(Q_m(0), P_m(0), I_m(0))$ (cf condition 1(d) of definition 2 2, and the second remark following; the representations of w_1 considered here are column finite). The vectors $\Phi_m(\zeta, \theta)$, $A(Q_m(0), P_m(0), I_m(0))\Phi_m(\zeta, \theta)$ are associated with elements of $\mathcal{T}_m$, while the vectors $\Phi_\infty(\zeta, \theta)$, $A(Q_\infty(0), P_\infty(0), I_\infty(0))\Phi_\infty(\zeta, \theta)$ are associated with elements of $\mathcal{T}_\infty$.

The next step needed in the argument is to show that (40) can be extended to times $t > 0$. This is difficult to establish for general polynomial Hamiltonians, even in one dimension Let $Q_m(t)$, $P_m(t)$ be the solution of the operator differential equations (36) with the initial condition $Q_m(0)$, $P_m(0)$ as given in (33) For $r, s \geqslant -M_m$, let

$$[Q_m(t)]_{rs} = (\psi_{(m)r}, Q_m(t)\psi_{(m)s})_m \qquad [P_m(t)]_{rs} = (\psi_{(m)r}, P_m(t)\psi_{(m)s})_m \tag{42}$$

denote the matrix elements of $Q_m(t)$ and $P_m(t)$ in the $\psi_{(m)r}$ basis Recall that $Q_\infty(0)$ and $P_\infty(0)$ have already been defined as the limits of $Q_m(0)$ and $P_m(0)$, in the sense that

$$\begin{aligned}
[Q_\infty(0)]_{rs} = \left(\psi_{(\infty)r}, Q_\infty(0)\psi_{(\infty)s}\right)_\infty &= \lim_{m\to\infty} \left(\psi_{(m)r}, Q_m(0)\psi_{(m)s}\right)_m \\
&= \lim_{m\to\infty} [Q_m(0)]_{rs} \\
[P_\infty(0)]_{rs} = \left(\psi_{(\infty)r}, P_\infty(0)\psi_{(\infty)s}\right)_\infty &= \lim_{m\to\infty} \left(\psi_{(m)r}, P_m(0)\psi_{(m)s}\right)_m \\
&= \lim_{m\to\infty} [P_m(0)]_{rs}
\end{aligned} \tag{43}$$

for all r, s Define $Q_\infty(t)$ and $P_\infty(t)$ similarly in terms of the limits

$$\begin{aligned}
[Q_\infty(t)]_{rs} = \left(\psi_{(\infty)r}, Q_\infty(t)\psi_{(\infty)s}\right)_\infty &= \lim_{m\to\infty} \left(\psi_{(m)r}, Q_m(t)\psi_{(m)s}\right)_m \\
&= \lim_{m\to\infty} [Q_m(t)]_{rs} \\
[P_\infty(t)]_{rs} = \left(\psi_{(\infty)r}, P_\infty(t)\psi_{(\infty)s}\right)_\infty &= \lim_{m\to\infty} \left(\psi_{(m)r}, P_m(t)\psi_{(m)s}\right)_m \\
&= \lim_{m\to\infty} [P_m(t)]_{rs}
\end{aligned} \tag{44}$$

meaning pointwise convergence in t for all integers r, s

It is conjectured that the connection between the quantum and classical dynamics is then established through the extension of (40) to times $t > 0$, so that

$$\begin{aligned}
&\lim_{m\to\infty} Q_m(t)\Phi_m(\zeta,\theta) = Q_\infty(t)\Phi_\infty(\zeta,\theta) = q(t)\Phi_\infty(\zeta,\theta) \\
&\lim_{m\to\infty} P_m(t)\Phi_m(\zeta,\theta) = P_\infty(t)\Phi_\infty(\zeta,\theta) = p(t)\Phi_\infty(\zeta,\theta) \\
&\lim_{m\to\infty} I_m(t)\Phi_m(\zeta,\theta) = I_\infty(t)\Phi_\infty(\zeta,\theta) = 1\Phi_\infty(\zeta,\theta)
\end{aligned} \tag{45}$$

where $(q(t), p(t))$ is the classical trajectory having initial value (q_0, p_0). Then the generalized one-dimensional subspace spanned by $\Phi_\infty(\zeta,\theta)$ will be invariant under the action of $Q_\infty(t)$, $P_\infty(t)$ and $I_\infty(t)$ in the contraction limit, and will be an eigenspace of those operators, corresponding to eigenvalues on a classical trajectory.

Key steps in this argument that have yet to be established for general Hamiltonians involve showing that (a) the vectors $\psi_{(m)r}$ lie in the domains of $Q_m(t)$ and $P_m(t)$, so that the matrix elements (42) of these operators are well defined in this basis; (b) the limits (44) exist, and (c) the results (45) follow.

Rather than exploring these difficult problems for general Hamiltonians at this stage [10], simple systems for which the idea can be pushed through completely are considered as support for the conjecture.

4.1. Systems with linear dynamics

It is easy to establish the desired results for Hamiltonians (28) with $n \leqslant 2$, which lead to linear dynamics. For example, consider the simple harmonic oscillator Hamiltonian

$$H(Q,P) = P^2/(2M) + (1/2)M\omega^2 Q^2 \tag{46}$$

where M is the mass of the oscillator and ω is the angular frequency of oscillation The classical Hamiltonian function is

$$H(q,p) = p^2/(2M) + (1/2)M\omega^2 q^2 \tag{47}$$

from which the classical dynamical equations are $q = p/M$ and $\dot{p} = -M\omega^2 q$, with general solutions of the form

$$q(t) = q_0 \cos\omega t + \frac{p_0}{M\omega}\sin\omega t \qquad p(t) = p_0\cos\omega t - M\omega q_0 \sin\omega t \tag{48}$$

where $(q_0, p_0) = (q(0), p(0))$ is the initial condition of the classical trajectory

For each m

$$H(Q_m, P_m) = P_m^2/2M + (1/2)M\omega^2 Q_m^2. \tag{49}$$

Consequently

$$\dot{Q}_m(t) = P_m(t)/M \qquad \dot{P}_m(t) = -M\omega^2 Q_m(t) \tag{50}$$

which have general solutions

$$Q_m(t) = Q_m(0)\cos\omega t + \frac{P_m(0)}{M\omega}\sin\omega t \qquad P_m(t) = P_m(0)\cos\omega t - M\omega Q_m(0)\sin\omega t \tag{51}$$

Using (33) and (21), these equations converge in the limit $m \to \infty$, in the sense of (43) and (44), to

$$Q_\infty(t) = Q_\infty(0)\cos\omega t + \frac{P_\infty(0)}{M\omega}\sin\omega t \qquad P_\infty(t) = P_\infty(0)\cos\omega t - M\omega Q_\infty(0)\sin\omega t. \tag{52}$$

It then follows that $Q_\infty(t)\Phi_\infty(\zeta,\theta) = q(t)\Phi_\infty(\zeta,\theta)$ and $P_\infty(t)\Phi_\infty(\zeta,\theta) = p(t)\Phi_\infty(\zeta,\theta)$ as in (45), giving the conjectured behaviour.

4.2. A non-linear oscillator

This system has been discussed by Milburn [7], Yurke and Stoler [8] and Daniel and Milburn [9], as a model non-linear system which is exactly soluble, but whose quantum dynamics shows, at least for sufficiently large times, a departure from the behaviour expected classically. Systems of this type have also been discussed by Berry [11]. It is therefore of interest to see that the present approach yields the classical dynamics in the contraction limit. Let $H_0(Q,P) = P^2/2M + (1/2)M\omega^2 Q^2$ be the simple harmonic oscillator Hamiltonian operator described above and let the Hamiltonian operator for the system be $H(Q,P) = H_0(Q,P) + \mu H_0(Q,P)^2$, where μ is a constant with dimensions of $[\text{energy}]^{-1}$. The Hamiltonian function for the corresponding classical system is

$$H(q,p) = H_0(q,p) + \mu H_0(q,p)^2 = \left(\frac{p^2}{2M} + \tfrac{1}{2}M\omega^2 q^2\right) + \mu\left(\frac{p^2}{2M} + \tfrac{1}{2}M\omega^2 q^2\right)^2 \tag{53}$$

and the corresponding classical dynamical equations are

$$\dot q = [1 + 2\mu H_0(q,p)]p/M = \Omega p/M \qquad \dot p = -M\omega^2[1 + 2\mu H_0(q,p)]q = -M\omega^2\Omega q \tag{54}$$

where

$$\Omega = 1 + 2\mu H_0(q,p) \equiv 1 + \mu[p_0^2/M + M\omega^2 q_0^2]. \tag{55}$$

Alternatively, $\ddot q = -\omega^2\Omega^2 q$, $\ddot p = -\omega^2\Omega^2 p$. Since H_0 and Ω are constants of the motion, these equations have general solutions of the form

$$q(t) = q_0 \cos\omega\Omega t + \frac{p_0}{M\omega}\sin\omega\Omega t \qquad p(t) = p_0\cos\omega\Omega t - M\omega q_0 \sin\omega\Omega t \tag{56}$$

where the frequency of oscillation $\omega\Omega$ of the non-linear oscillator is explicitly energy dependent.

Suppose for convenience that $\lambda = \sqrt{\hbar/M\omega}$ and $\kappa = \sqrt{\hbar M\omega}$, and set $E = \hbar\omega$. Then for each m

$$\begin{aligned} H(Q_m,P_m) &= H_0(Q_m,P_m) + \mu H_0(Q_m,P_m)^2 \\ &= E\left[(P_m^2/2\kappa^2 + Q_m^2/2\lambda^2) + \mu E\left(P_m^2/2\kappa^2 + Q_m^2/2\lambda^2\right)^2\right]. \end{aligned} \tag{57}$$

Consequently, for each m, the quantum operator equations of motion are

$$\begin{aligned} \dot Q_m &= \frac{E}{\kappa^2}[P_m + \mu(H_0(Q_m,P_m)P_m + P_m H_0(Q_m,P_m))] \\ &= \frac{1}{M}[P_m(1 + 2\mu H_0(Q_m,P_m)) + \mathrm{i}\mu E\epsilon_m^2 M\omega Q_m] \\ \dot P_m &= -\frac{E}{\lambda^2}[Q_m + \mu(H_0(Q_m,P_m)Q_m + Q_m H_0(Q_m,P_m))] \\ &= -M\omega^2[Q_m(1 + 2\mu H_0(Q_m,P_m)) - \mathrm{i}\mu E\epsilon_m^2 P_m/M\omega] \end{aligned} \tag{58}$$

where ϵ_m is the contraction parameter. Now

$$(\dot Q_m \pm \mathrm{i}\dot P_m/M\omega) = (Q_m \pm \mathrm{i}P_m/M\omega)[\mp\mathrm{i}\omega(\mathbb{I}_m + 2\mu H_0(Q_m,P_m)) + \mathrm{i}\mu E\epsilon_m^2\omega\mathbb{I}_m] \tag{59}$$

which has general solutions

$$(Q_m(t) \pm \mathrm{i}P_m(t)/M\omega) = (Q_m(0) \pm \mathrm{i}P_m(0)/M\omega)\mathrm{e}^{\mp\mathrm{i}\omega(\mathbb{I}_m + 2\mu H_0)t + \mathrm{i}\mu E\epsilon_m^2\omega t\mathbb{I}_m}. \tag{60}$$

Since

$$\left(\psi_{(m)r}, \mathrm{e}^{\mp\mathrm{i}\omega(\mathbb{I}_m + 2\mu H_0)t}\psi_{(m)s}\right)_m = \delta_{rs}\mathrm{e}^{\mp\mathrm{i}\omega[1 + \mu E(2M_m + 2r + 1)/m]t} \tag{61}$$

which approaches $\delta_{r,s}e^{\mp i\omega(1+2\mu E\zeta)t}$ as $m \to \infty$, then in the contraction limit [12]

$$e^{\mp i\omega(I_m+2\mu H_0)t+i\mu E\epsilon_m^2\omega t I_m} \to e^{\mp i\omega(1+2\mu E\zeta)t}\mathbb{I}_\infty \tag{62}$$

and

$$(Q_\infty(t) \pm iP_\infty(t)/M\omega) = (Q_\infty(0) \pm iP_\infty(0)/M\omega)e^{\mp i\omega(1+2\mu E\zeta)t}. \tag{63}$$

But $1 + 2\mu E\zeta = 1 + 2\mu H_0(q_0, p_0) = \Omega$, so

$$\begin{aligned} Q_\infty(t) &= Q_\infty(0)\cos\omega\Omega t + \frac{1}{M\omega}P_\infty(0)\sin\omega\Omega t \\ P_\infty(t) &= P_\infty(0)\cos\omega\Omega t - M\omega Q_\infty(0)\sin\omega\Omega t. \end{aligned} \tag{64}$$

As in the previous two examples, it is immediately seen that $Q_\infty(t)\Phi_\infty(\zeta,\theta) = q(t)\Phi_\infty(\zeta,\theta)$ and $P_\infty(t)\Phi_\infty(\zeta,\theta) = p(t)\Phi_\infty(\zeta,\theta)$.

A feature of these examples is that the functions $Q_m(t)$ and $P_m(t)$ exhibit a simple dependence on $Q_m(0)$, $P_m(0)$ and t which facilitates computation of their behaviour in the contraction limit, even though the last example has an added dependence on the operator H_0. In fact, any quantum mechanical system with a single degree of freedom whose operators $Q(t)$, $P(t)$ in the Heisenberg picture are polynomial in $Q(0)$, $P(0)$ and analytic in t, can obviously be treated directly in this manner. More importantly, it can be seen explicitly that (45) holds in each case, so that the one-dimensional generalized subspace spanned by $\Phi_\infty(\zeta,\theta)$ remains invariant under the action of these operators, and is an eigenspace of $(Q_\infty(t), P_\infty(t))$ corresponding to eigenvalues on a trajectory of the classical system. Furthermore,

$$\lim_{m\to\infty} A(Q_m(t), P_m(t))\Phi_m(\zeta,\theta) = A(Q_\infty(t), P_\infty(t))\Phi_\infty(\zeta,\theta) = A(q(t), p(t))\Phi_\infty(\zeta,\theta) \tag{65}$$

for arbitrary polynomials $A(Q,P)$ A particular case of interest is when $A(Q,P)$ is the Hamiltonian operator $H(Q,P)$, and $A(q(t),p(t))$ is the classical energy. It can also be seen that for each example

$$\begin{aligned} \lim_{m\to\infty} \dot{Q}_m(t)\Phi_m(\zeta,\theta) &= \lim_{m\to\infty} \frac{1}{i\epsilon_m^2\hbar}[Q_m, H(Q_m,P_m)]\Phi_m(\zeta,\theta) \\ &= \dot{Q}_\infty(t)\Phi_\infty(\zeta,\theta) \\ &= \{q, H(q,p)\}\Phi_\infty(\zeta,\theta) \\ \lim_{m\to\infty} \dot{P}_m(t)\Phi_m(\zeta,\theta) &= \lim_{m\to\infty} \frac{1}{i\epsilon_m^2\hbar}[P_m, H(Q_m,P_m)]\Phi_m(\zeta,\theta) \\ &= \dot{P}_\infty(t)\Phi_\infty(\zeta,\theta) \\ &= \{p, H(q,p)\}\Phi_\infty(\zeta,\theta) \end{aligned} \tag{66}$$

where $\{\cdot,\cdot\}$ is the classical Poisson bracket.

If $(q_1(0), p_1(0))$ and $(q_2(0), p_2(0))$ are two initial conditions of the classical system corresponding to the same ζ value but possibly different θ values, θ_1 and θ_2, then

$$\lim_{m\to\infty} Q_m(t)\Phi_m(\zeta,\theta_i) = q_i(t)\Phi_\infty(\zeta,\theta_i) \qquad \lim_{m\to\infty} P_m(t)\Phi_m(\zeta,\theta_i) = p_i(t)\Phi_\infty(\zeta,\theta_i) \tag{67}$$

for $i = 1, 2$. Furthermore

$$\begin{aligned}(\Phi_\infty(\zeta,\theta_1), Q_\infty(t)\Phi_\infty(\zeta,\theta_2))_\infty &= q_2(t)\delta(\theta_2-\theta_1) = q_1(t)\delta(\theta_2-\theta_1)\\ (\Phi_\infty(\zeta,\theta_1), P_\infty(t)\Phi_\infty(\zeta,\theta_2))_\infty &= p_2(t)\delta(\theta_2-\theta_1) = p_1(t)\delta(\theta_2-\theta_1)\end{aligned} \tag{68}$$

and indeed

$$(\Phi_\infty(\zeta,\theta_1), A(Q_\infty(t), P_\infty(t))\Phi_\infty(\zeta,\theta_2))_\infty = A(q_1(t), p_1(t))\delta(\theta_2-\theta_1) \tag{69}$$

for any finite polynomial $A(Q_\infty(t), P_\infty(t))$.

A super-selection rule is a statement that rules out certain vectors in the Hilbert space associated with a quantum system as not corresponding to physically realisable states [13]. In this context, the linear superposition $(a_1\chi_1 + a_2\chi_2)$, of two different physically realisable states χ_1 and χ_2 would be considered physically unrealisable if there exists no observable A such that $(\chi_1, A\chi_2) \neq 0$.

For the model one-dimensional systems described above, suppose that the only observables correspond to elements of the enveloping algebra generated by Q, P and I; this enveloping algebra contains H which is polynomial in Q and P by hypothesis. As has been shown above, the representations of the enveloping algebra of w_1 generated by $Q_m(0)$, $P_m(0)$ and $I_m(0)$ are contracted to the representation of the enveloping algebra of t_3 generated by $Q_\infty(0)$, $P_\infty(0)$ and $I_\infty(0)$ Furthermore, it follows from (65) and (66) that each of the generalized one-dimensional subspaces associated with the irreducible representations of t_3 obtained in the contraction limit, is stable under the action of the dynamics and of the enveloping algebra generated by the observables $Q_\infty(t), P_\infty(t)$, and is associated with a classical trajectory.

It can now be seen in addition that there are in the classical (contraction) limit, natural superselection rules The vectors $\Phi_\infty(\zeta,\theta)$ can be taken to be physically realisable because they are eigenvectors of the contracted Hamiltonian operator H_∞, as a result of (40), but superpositions of the vectors $\Phi_\infty(\zeta,\theta)$ with different θ are unphysical as a result of (69) [14] Thus the Hilbert space structure becomes redundant in the contraction limit; the vector space $\mathcal{H}_\infty$ is effectively reduced to a union (rather than a direct sum) of generalized one-dimensional subspaces [5]

To summarize: between any two of these subspaces there is a superselection rule; each subspace remains invariant under the dynamics; and each is an eigenspace of $(Q_\infty(t), P_\infty(t))$ corresponding to eigenvalues $(q(t), p(t))$ on a classical trajectory.

5. Concluding remarks

For the simple systems considered in this paper, consideration of the classical limit in terms of a Lie algebra contraction as described above provides a new way of viewing this limiting process. A distinctive and attractive feature of this approach is that quantum mechanics and classical mechanics are treated in a unified way. In each case one has a representation of a kinematical algebra and its enveloping algebra in a Hilbert space In the classical case, this algebra is Abelian; the representation is a direct integral of one-dimensional irreducible Hermitian representations with a superselection rule between each of the corresponding one-dimensional subspaces. It is conjectured that similar results hold for an arbitrary polynomial Hamiltonian operator

$H(Q, P)$ but clearly, general technical conditions need to be investigated under which (45) is satisfied [10]. It is hoped that this will be the topic of future studies.

It should be remarked in closing that there is a natural and direct generalization of the ideas presented here to the classical limit of a quantum system with n degrees of freedom, involving the contraction [10] of representations of its kinematical algebra w_n to representations of the Abelian Lie algebra t_{2n+1}. For example, similar results to the above are obtained for the n-dimensional analogue of the non-linear system with Hamiltonian (53), so the approach is certainly not limited to one-dimensional systems Indeed, there is no difficulty in obtaining the generalization of (40) to the case of n degrees of freedom, whatever the Hamiltonian. However as noted in the introduction, the important question as to whether equations (45) hold for non-integrable systems remains unanswered.

Other features that can arise for systems with several degrees of freedom are accidental symmetry algebras and dynamical (spectrum-generating) algebras. The relationship between classical and quantum symmetries is not always obvious [15], and as suggested by a referee of the first version of this paper, it would therefore be interesting to examine the behaviour in the classical contraction limit of the representations of such algebras.

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Appendix

It could be expected from a consideration of (23) that there might be a sequence of w_1 representations which when contracted gives a representation of t_3 equivalent to a direct integral of one-dimensional Hermitian representations labelled (ζ, θ) for $0 \leqslant \zeta < \infty$, $0 \leqslant \theta < 2\pi$. This appears not to be possible. However a partial result in this direction can be obtained.

A construction is now given for the contraction of a sequence of infinite-dimensional Hermitian irreducible matrix representations of w_1 to an infinite-dimensional matrix representation of t_3 which is decomposable into a countably infinite direct sum of representations of t_3 of the type given in (11).

The positive rationals are ordered without repetition in some way so that all of them are included. For each positive integer n, a value of the contraction parameter $\epsilon_m = 1/\sqrt{m}$ is chosen so that a nucleus of $(2n+1)$ basis vectors can be established for each of the n subrepresentations of t_3 parameterized by the first n rationals. As n gets larger, the nuclei of more subrepresentations are included in the scheme for the basis corresponding to the particular value of the contraction parameter.

Let $\{r_t\}_{t=0}^{\infty}$ enumerate the positive rationals in some manner such that $r_t = r_{t'}$ $\Leftrightarrow t = t'$. Then for each $n > 1$, let δ be the positive rational number defined by $\delta = \min_{0 \leqslant t,t' \leqslant n} |r_t - r_{t'}|$ and choose m_n and ϵ_n satisfying $m_n \epsilon_n^2 = 1$ and $m_n < m_{n+1}$ so that $\min_{0 \leqslant t,t' \leqslant n} |[\![r_t m_n]\!] - [\![r_{t'} m_n]\!]| \geqslant 2n$ If $\delta = \delta_1/\delta_2$ for positive integers δ_1, δ_2, where $gcd(\delta_1, \delta_2) = 1$, then $m_n = 2(n+1)\delta_2$ is a sufficient although not 'minimal' choice.

Then let the first $(n+1)(n+2)/2$ basis vectors in the new basis be given by

$$\phi_{(m)p(p+1)/2+q} = \xi_{[r_{p-q}m_n]+s_q} \tag{A1}$$

for $p = 0,\ldots,n$; $q = 0, \ldots, p$, where $\{s_q\}_{q=0}^{\infty} = \{0,-1,1,-2,2,\ldots\}$ is the sequence with elements $s_{2(q-1)} = q-1$ and $s_{2q-1} = -q$ for positive integers q.

In the contraction limit $n \to \infty$, $(m_n \to \infty)$, the representation is decomposable into a countable infinity of t_3 representations of the type given in (16) with ζ replaced by r_t. Associated with each t_3 representation, there are common generalized position-momentum eigenvectors $\Phi(r_t,\theta)$ for which

$$\begin{aligned}
Q_\infty \Phi(r_t,\theta) &= \lambda\sqrt{2r_t}\cos\theta\,\Phi(r_t,\theta)\\
P_\infty \Phi(r_t,\theta) &= \kappa\sqrt{2r_t}\sin\theta\,\Phi(r_t,\theta)\\
I_\infty \Phi(r_t,\theta) &= \Phi(r_t,\theta)\\
\big(\Phi(r_{t'},\theta'),\Phi(r_t,\theta)\big)_\infty &= \delta_{tt'}\delta(\theta-\theta').
\end{aligned} \tag{A2}$$

Since each of the infinite-dimensional representations of t_3 in the direct sum is equivalent to a direct integral of one-dimensional irreducible matrix representations, the direct sum representation itself is equivalent to the representation

$$\begin{aligned}
Q_\infty &= \lambda \sum_{t=0}^{\infty}\oplus \int_0^{2\pi} \oplus[\sqrt{2r_t}\cos\theta_t]\,\mathrm{d}\theta_t\\
P_\infty &= \kappa \sum_{t=0}^{\infty}\oplus \int_0^{2\pi} \oplus[\sqrt{2r_t}\sin\theta_t]\,\mathrm{d}\theta_t\\
I_\infty &= \sum_{t=0}^{\infty}\oplus \int_0^{2\pi} \oplus[1]\,\mathrm{d}\theta_t.
\end{aligned} \tag{A3}$$

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[14] In the case where the sequence of u_1 representations is contracted to a direct sum of two or more t_3 representations of the type (21), each parameterized by a different ζ value, superpositions of the vectors $\Phi_\infty(\zeta,\theta)$ with different ζ and θ values are also unphysical.
[15] See for example, Dirl R and Moshinsky M 1985 *J Phys. A: Math. Gen.* **18** 2423